

# MUTATING BRAUER TREES

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**ABSTRACT.** In this paper we introduce mutation of Brauer trees. We show that our mutation of Brauer trees explicitly describes the tilting mutation of Brauer tree algebras introduced by Okuyama and Rickard.

Mutation plays an important role in representation theory, especially in tilting theory and cluster tilting theory. Cluster tilting theory deals with the combinatorial structure of 2-Calabi-Yau triangulated categories and is applied to categorification of Fomin-Zelevinsky cluster algebras. This is closely related with tilting theory of hereditary algebras, and the class of tilting complexes is one of the most important classes from Morita theoretic viewpoint [Ri2]. Now it is an important problem to study the combinatorial structure of tilting complexes for finite dimensional algebras. In this paper we consider this problem for Brauer tree algebras, which form one of the most basic classes of symmetric algebras. The main result of this paper is to describe explicitly the combinatorics of tilting mutation for Brauer tree algebras. This is given by mutation of Brauer trees, which is a new operation introduced in this paper.

In Section 1 we recall a construction of tilting complexes  $T$  of symmetric algebras  $A$  which we call *Okuyama-Rickard complexes*. It was introduced by Okuyama and Rickard, and has been played an important role in the study of Broué's abelian defect group conjecture. Since it is a special case of tilting mutation as we pointed out in [AI], we call the endomorphism algebra  $\text{End}_{\mathcal{K}^b(\text{proj-}A)}(T)$  *tilting mutation* of  $A$ .

In Section 2 we introduce a combinatorial operation of Brauer trees which we call *mutation of Brauer trees*. Our main result shows that tilting mutation of Brauer tree algebras which we explained above is compatible with our mutation of Brauer trees. A special case was given in [KZ] and [Z] (see Remark 2.3). As an application of our main result, we give braid-type relations for tilting mutation of Brauer tree algebras (Theorem 2.5).

In Section 3 we prove our main result. Our proof is very simple and seems to be interesting by itself.

## 1. PRELIMINARY

Through this paper, let  $A$  be a finite dimensional  $k$ -algebra for an algebraically closed field  $k$  and we assume that  $A$  is basic and indecomposable as an  $A$ - $A$ -bimodule. Let  $\{e_1, e_2, \dots, e_n\}$  be a basic set of orthogonal local idempotents in  $A$  and put  $E = \{1, 2, \dots, n\}$ . For each  $i \in E$ , we set  $P_i = e_i A$  and  $S_i = P_i / \text{rad} P_i$ .

We denote by  $\text{mod-}A$  the category of finitely generated right  $A$ -modules, by  $\text{proj-}A$  the full subcategory of  $\text{mod-}A$  consisting of finitely generated projective right  $A$ -modules, by

$\underline{\text{mod}}\text{-}A$  the stable module category of  $\text{mod}\text{-}A$  and by  $\mathbf{K}^b(\text{proj}\text{-}A)$  the homotopy category of bounded complexes over  $\text{proj}\text{-}A$ .

Let us start with recalling the complex introduced by Okuyama and Rickard [Ok, Ri], which is a special case of silting mutation defined in [AI].

**Definition 1.1.** Let  $E_0$  be a subset of  $E$  and put  $e = \sum_{i \in E_0} e_i$ . For any  $i \in E$ , we define a complex by

$$T_i = \begin{cases} \begin{array}{ccc} (0\text{th}) & & (1\text{st}) \\ & & \\ P_i & \longrightarrow & 0 \end{array} & (i \in E_0) \\ \begin{array}{ccc} Q_i & \xrightarrow{\pi_i} & P_i \end{array} & (i \notin E_0) \end{cases}$$

where  $Q_i \xrightarrow{\pi_i} P_i$  is a minimal projective presentation of  $e_i A / e_i A e A$ . Now we define  $T := T(E_0) := \bigoplus_{i \in E} T_i$  and call it the *Okuyama-Rickard complex with respect to  $E_0$* .

The following observation shows the importance of Okuyama-Rickard complexes.

**Proposition 1.2.** [Ok, Proposition 1.1] *If  $A$  is a symmetric algebra, then any Okuyama-Rickard complex  $T$  is tilting. In particular  $\text{End}_{\mathbf{K}^b(\text{proj}\text{-}A)}(T)$  is derived equivalent to  $A$ .*

If we drop the assumption that  $A$  is symmetric, an Okuyama-Rickard complex is not necessarily a tilting complex but still a silting complex.

**Definition 1.3.** Let  $B$  be a finite dimensional  $k$ -algebra. For any  $i \in E$ , we say that  $B$  is the *tilting mutation of  $A$  with respect to  $i$*  if  $B \simeq \text{End}_{\mathbf{K}^b(\text{proj}\text{-}A)}(T(E \setminus \{i\}))$  and write  $A \xrightarrow{i} B$  or  $B = \mu_i(A)$ .

The aim of this paper is to introduce mutation of Brauer trees and study the relationship with tilting mutation of Brauer tree algebras.

Let us recall the definitions of Brauer trees and Brauer tree algebras

**Definition 1.4.** [Alp, GR] A *Brauer graph*  $G$  is a finite connected graph, together with the following data:

- (i) There exists a cyclic ordering of the edges adjacent to each vertex, usually described by the clockwise ordering given by a fixed planar representation of  $G$ ;
- (ii) For each vertex  $v$ , there exists a positive integer  $m_v$  assigned to  $v$ , called the *multiplicity*. We call a vertex  $v$  *exceptional* if  $m_v > 1$ .

A *Brauer tree*  $G$  is a Brauer graph which is a tree and having at most one exceptional vertex.

A *Brauer tree algebra*  $A = A_G$  is a basic algebra given by a Brauer tree  $G$  as follows:

- (i) There exists a one-to-one correspondence between simple  $A$ -modules  $S_i$  and edges  $i$  of  $G$ ;
- (ii) For any edge  $i$  of  $G$ , the projective indecomposable  $A$ -module  $P_i$  has  $\text{soc}(P_i) \simeq P_i / \text{rad}(P_i)$  and  $\text{rad}(P_i) / \text{soc}(P_i)$  is the direct sum of two uniserial modules whose composition factors are, for the cyclic ordering  $(i, i_1, \dots, i_a, i)$  of the edges adjacent to a vertex  $v$ ,  $S_{i_1}, \dots, S_{i_a}, S_i, S_{i_1}, \dots, S_{i_a}$  (from the top to the socle) where  $S_i$  appears  $m_v - 1$  times.

Note that  $A_G$  is uniquely determined by  $G$  up to isomorphism. Moreover it is a symmetric algebra.

We say that a Brauer tree  $G$  is a *star* if there exists a vertex  $v$  of  $G$  such that all edges of  $G$  appear in the cyclic ordering adjacent to  $v$ .

The following well-known result shows that derived equivalence classes of Brauer tree algebras are determined by certain numerical invariants.

**Theorem 1.5.** [Ri, Theorem 4.2]

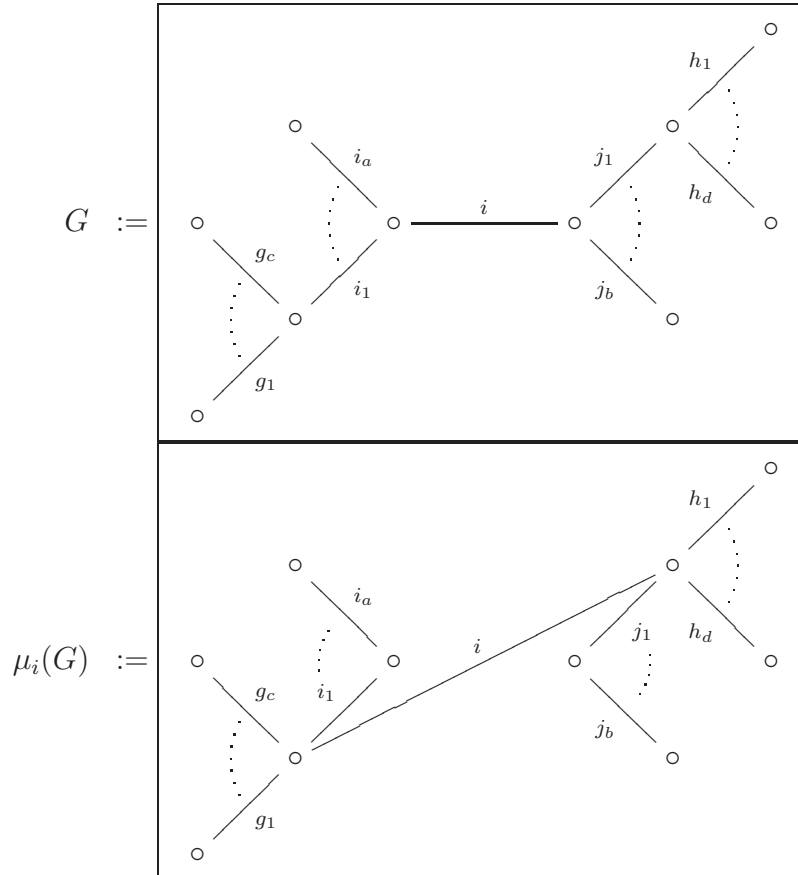
- (1) For any Brauer tree algebra  $A$ , there exists a tilting complex  $P \in \mathbf{K}^b(\text{proj-}A)$  such that the endomorphism algebra  $\text{End}_{\mathbf{K}^b(\text{proj-}A)}(P)$  of  $P$  is a Brauer tree algebra for a star with exceptional vertex in the center if it exists.
- (2) In particular, derived equivalence classes of Brauer tree algebras are determined by the number of the edges and the multiplicities of the vertices.

## 2. MUTATING BRAUER TREES

Let us start with the definition of mutation of Brauer trees.

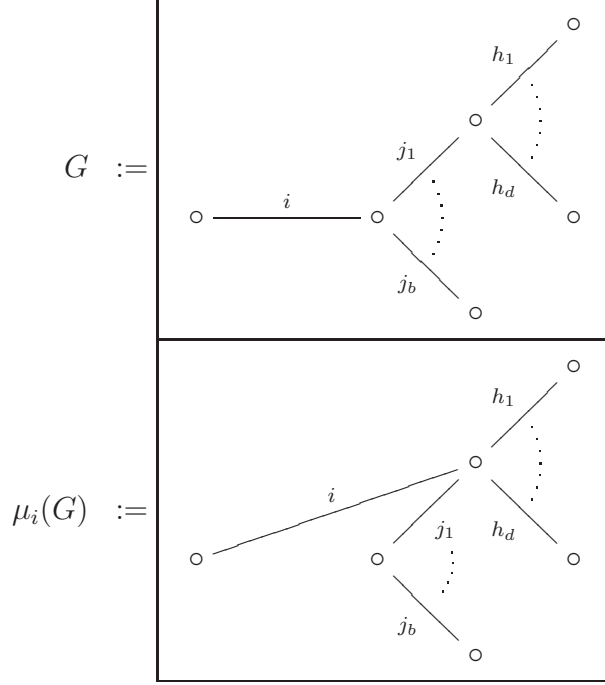
**Definition 2.1.** Let  $G$  be a Brauer tree and  $i$  be an edge of  $G$ . We define a Brauer tree  $\mu_i(G)$  which is called *mutation* of  $G$  with respect to  $i$  as follows.

- (1) The edge  $i$  is an internal edge: Let  $(i, i_1, \dots, i_a, i)$  and  $(i, j_1, \dots, j_b, i)$  be the cyclic orderings containing  $i$ . Let  $(i_1, g_1, \dots, g_c, i_1)$  and  $(j_1, h_1, \dots, h_d, j_1)$  be the cyclic orderings containing  $i_1$  and  $j_1$ , which do not contain  $i$ .



The multiplicities of vertices do not change.

- (2) The edge  $i$  is an external edge: Let  $(i, j_1, \dots, j_b, i)$  be the cyclic ordering containing  $i$ . Let  $(j_1, h_1, \dots, h_d, j_1)$  be the cyclic ordering containing  $j_1$ , which does not contain  $i$ .



The multiplicities of vertices do not change.

Now we state the main theorem in this paper.

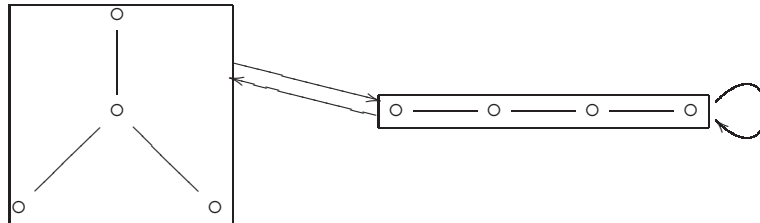
**Theorem 2.2.** *For any Brauer tree  $G$  and any edge  $i$  of  $G$ , we have an isomorphism  $\mu_i(A_G) \simeq A_{\mu_i(G)}$  of  $k$ -algebras, sending each idempotent  $e_j$  to  $e_j$ .*

We prove the theorem above in the next section. This can be regarded as an analogue of derived equivalences associated with Bernstein-Gelfand-Ponomarev reflection of quivers [BGP] and Derksen-Weyman-Zelevinsky mutation of quivers with potentials [DWZ, BIRS].

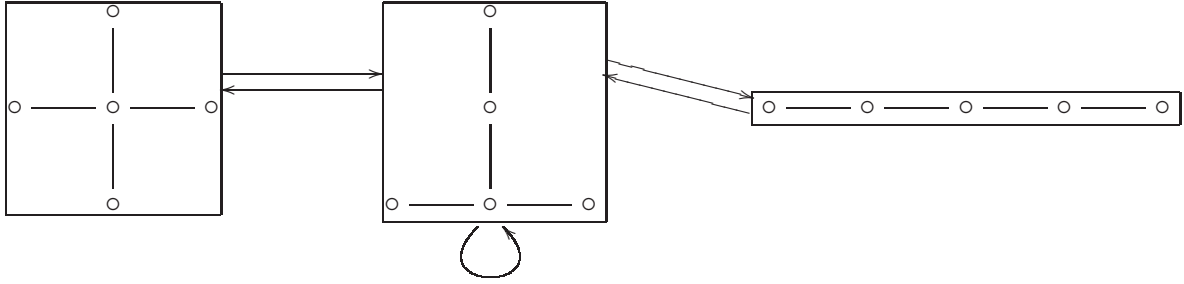
**Remark 2.3.** Note that a special case of Theorem 2.2 was given in [KZ] and [Z], where they only considered the case (2) in Definition 2.1 such that there are no exceptional vertices.

**Example 2.4.** We give graphs of mutation of Brauer trees.

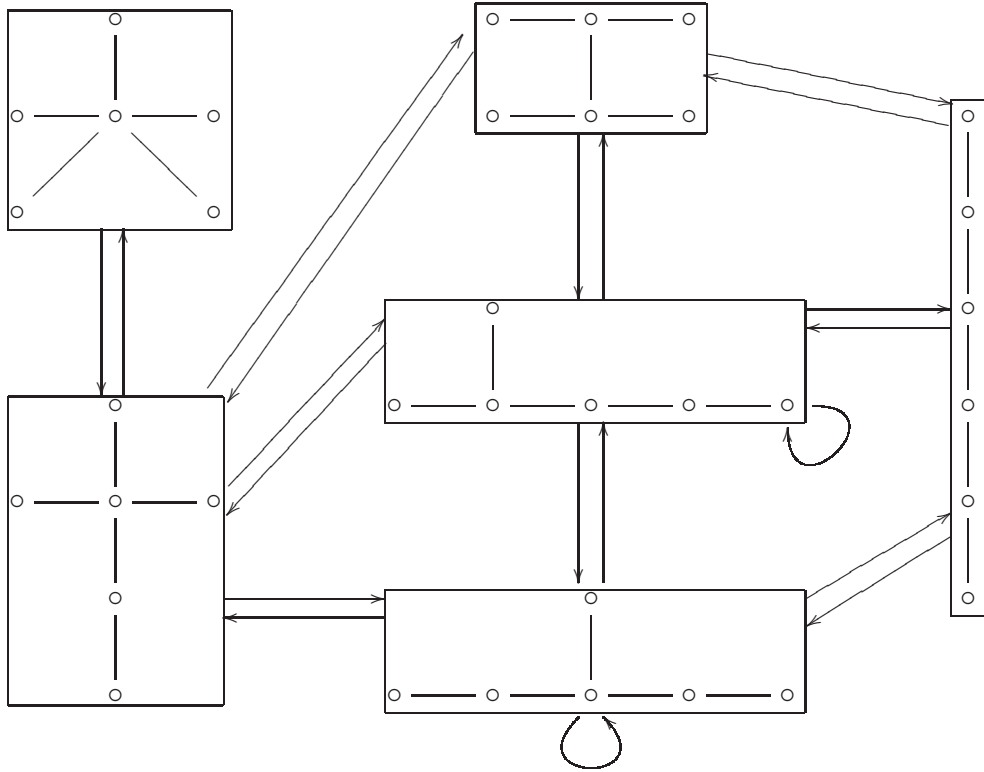
- (1) Brauer trees with 3 edges and without exceptional vertices:



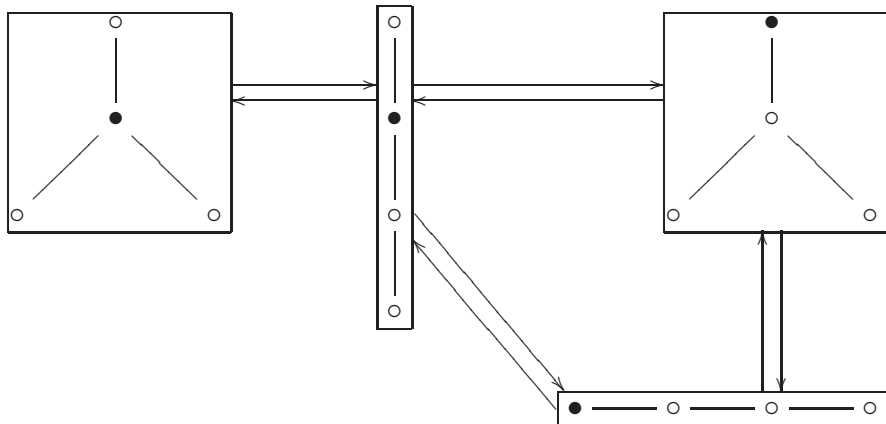
(2) Brauer trees with 4 edges and without exceptional vertices:



(3) Brauer trees with 5 edges and without exceptional vertices:



(4) Brauer trees with 3 edges and exceptional vertex  $\bullet$ :



We give some applications of Theorem 2.2.

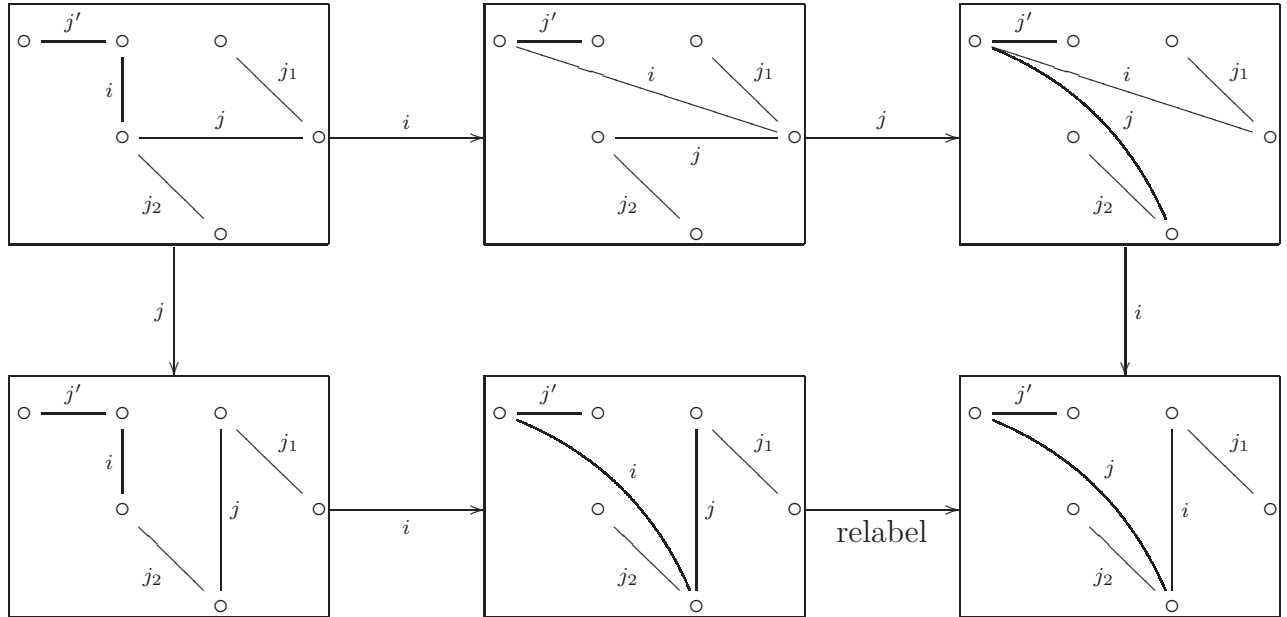
For edges  $i$  and  $j$  in a Brauer tree we say that  $j$  *follows*  $i$  if there exists a cyclic ordering of the form  $(\dots, i, j, \dots)$ .

**Theorem 2.5.** *Let  $A$  be a Brauer tree algebra. Then for any edges  $i, j$  of the Brauer tree of  $A$ , we have the following relations:*

- (1)  $(\mu_i)^s(A) \simeq A$  for some positive integer  $s$ ;
- (2)  $\mu_j \mu_i(A) \simeq \mu_i \mu_j(A)$  if  $i$  does not follow  $j$  and  $j$  does not follow  $i$ ;
- (3)  $\mu_i \mu_j \mu_i(A) \simeq \mu_i \mu_j(A)$  if  $i$  does not follow  $j$  and  $j$  follows  $i$ ;
- (4)  $\mu_i \mu_j \mu_i(A) \simeq \mu_j \mu_i \mu_j(A)$  if  $i$  follows  $j$  and  $j$  follows  $i$ .

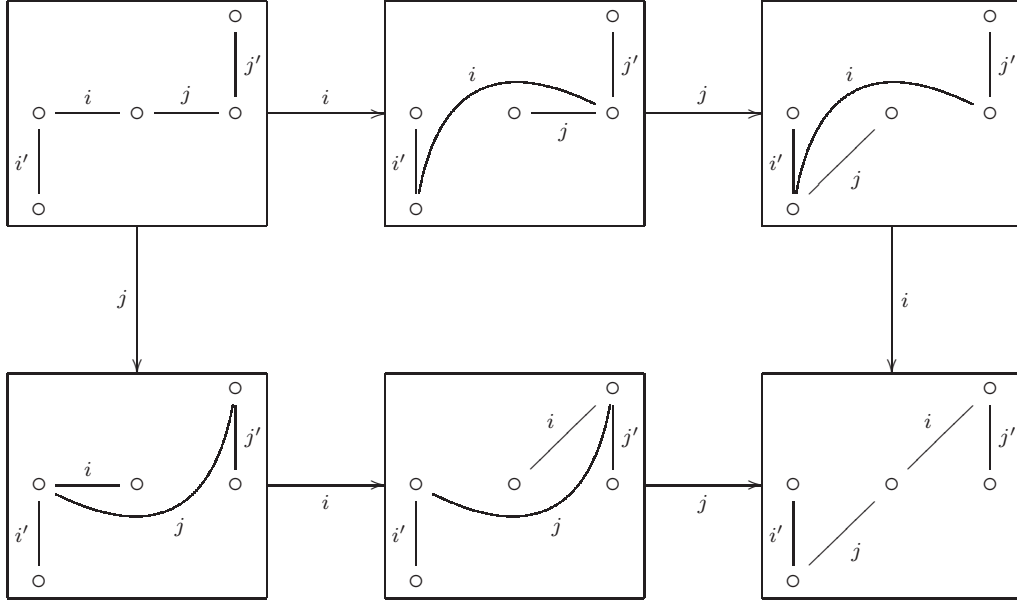
*Proof.* Let  $G$  be the Brauer tree of  $A$ . By Theorem 2.2, we only have to show the corresponding isomorphisms of Brauer trees for  $G$ .

- (1) We denote by  $\text{Br}(n, m)$  the set of labeled Brauer trees which have  $n$  edges and multiplicity  $m$  of the exceptional vertex. We can regard mutation as group action to  $\text{Br}(n, m)$ . Since  $\text{Br}(n, m)$  is a finite set, the order of mutation is finite.
- (2) This assertion can be checked easily.
- (3) Let  $j'$  follow  $i$  being not  $j$  and  $j_1, j_2$  follow  $j$ . We have the following mutation:



Hence we obtain  $\mu_i \mu_j \mu_i(G) \simeq \mu_i \mu_j(G)$ .

(4) Let  $i'$  follow  $i$  being not  $j$  and  $j'$  follow  $j$  being not  $i$ . We have the following mutation:



Hence one gets  $\mu_i \mu_j \mu_i(G) \simeq \mu_j \mu_i \mu_j(G)$ .

□

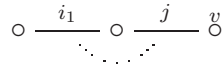
As another application of Theorem 2.2, we give a simple proof of the following stronger statement than Theorem 1.5.

**Corollary 2.6.** *Let  $G$  be a Brauer tree and  $\ell \geq 1$ . For every vertex  $v$  of  $G$  there exists a tilting complex  $T \in \mathbf{K}^b(\text{proj-}A_G)$  of the form  $(\cdots \rightarrow 0 \rightarrow P^0 \rightarrow P^1 \rightarrow 0 \rightarrow \cdots)$  with  $P^0, P^1 \in \text{proj-}A_G$  such that the Brauer tree of  $\text{End}_{\mathbf{K}^b(\text{proj-}A)}(T)$  is a star with the vertex  $v$  in the center.*

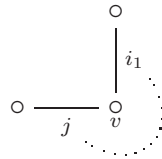
To prove the corollary above, we need the following preliminary results.

**Lemma 2.7.** *For any Brauer tree  $G$  and any vertex  $v$  of  $G$ , there exists a sequence  $i_1, \dots, i_\ell$  of distinct edges such that  $\mu_{i_\ell} \cdots \mu_{i_1}(G)$  is a star with the vertex  $v$  in the center.*

*Proof.* Take an edge  $i_1$  which is followed by an edge  $j$  with the vertex  $v$ :



By Theorem 2.2, we see that the Brauer tree  $\mu_{i_1}(G)$  is of the form



which says that the edge  $i_1$  has the vertex  $v$ . Continuing the argument, we obtain a star with the vertex  $v$  in the center. □

**Lemma 2.8.** *Let  $G$  be a Brauer tree and  $\ell \geq 1$ . If  $i_1, \dots, i_\ell$  are distinct edges of  $G$ , then there exists a tilting complex  $T \in \mathbf{K}^b(\text{proj-}A_G)$  of the form  $(\cdots \rightarrow 0 \rightarrow P^0 \rightarrow P^1 \rightarrow 0 \rightarrow \cdots)$  with  $P^0, P^1 \in \text{proj-}A_G$  such that  $\mu_{i_\ell} \cdots \mu_{i_1}(A_G) \simeq \text{End}_{\mathbf{K}^b(\text{proj-}A_G)}(T)$ .*

*Proof.* For any edge  $i$  of  $G$  we have a derived equivalence  $F_i : \mathbf{K}^b(\text{proj-}\mu_i(A_G)) \xrightarrow{\sim} \mathbf{K}^b(\text{proj-}A_G)$  given by the Okuyama-Rickard complex  $T(E \setminus \{i\})$  of  $A_G$ . Put  $T_\ell := F_{i_1} \cdots F_{i_\ell}(\mu_{i_\ell} \cdots \mu_{i_1}(A_G))$ , which is a tilting complex in  $\mathbf{K}^b(\text{proj-}A_G)$ . We show that  $T_\ell$  has a form

$$T_\ell = \left\{ \begin{array}{ccc} & (0\text{th}) & (1\text{st}) \\ & P \longrightarrow & 0 \\ \oplus & & \\ Q^0 \longrightarrow & & Q^1 \end{array} \right. \quad (2.8.1)$$

where  $P = \bigoplus_{j \in E \setminus \{i_1, \dots, i_\ell\}} P_j$  and  $Q^0, Q^1 \in \text{proj-}A_G$ . We use induction on  $\ell \geq 1$ . If  $\ell = 1$ , then one observes  $T_1 = T(E \setminus \{i_1\})$ . This says that  $T_1$  is of the form (2.8.1). Assume  $\ell \geq 2$ . It follows from the induction hypothesis that  $P_{i_\ell}$  is a direct summand of  $T_{\ell-1}$  and the complement  $T_{\ell-1} \setminus P_{i_\ell}$  is of the form  $R := (\cdots \rightarrow 0 \rightarrow R^0 \rightarrow R^1 \rightarrow 0 \rightarrow \cdots)$ . We see that  $T_\ell$  is given by the direct sum of  $R$  and a complex  $P'$  which admits a triangle  $P' \rightarrow R \rightarrow P_{i_\ell} \rightarrow P'[1]$ : see [AI]. Since  $P_{i_\ell}$  and  $R$  concentrate on degree 0 and  $(0, 1)$  respectively, one obtains that  $P'$  is of the form  $(\cdots \rightarrow 0 \rightarrow (P')^0 \rightarrow (P')^1 \rightarrow 0 \rightarrow \cdots)$ . This implies that  $T_\ell$  is of the form (2.8.1).  $\square$

Now Corollary 2.6 is an immediate consequence of Lemma 2.7 and Lemma 2.8.  $\square$

The following result is a direct consequence of Corollary 2.6.

**Corollary 2.9.** *Let  $A$  be a Brauer tree algebra. Any basic algebra which is derived equivalent to  $A$  is obtained from  $A$  by iterated tilting mutation.*

More generally, the statement above is shown for representation-finite symmetric algebras in [A].

### 3. PROOF OF MAIN THEOREM

In this section we prove the main theorem of this paper.

We define an  $(n \times n)$ -matrix  $C^A$  as  $C_{ij}^A = \dim_k \text{Hom}_A(P_i, P_j)$  for any  $i, j \in E$ , called the *Cartan matrix* of  $A$ . Note that if  $A$  is a symmetric algebra, then we have  $C_{ij}^A = C_{ji}^A$  for any  $i, j \in E$ .

We have the following property.

**Lemma 3.1.** *Let  $G$  be a Brauer tree. Then the Cartan matrix  $C^{A_G}$  of  $A_G$  is determined as follows:*

$$C_{ij}^{A_G} = \begin{cases} m_v + m_u & \text{if } i = j \text{ and the ends of } i \text{ are } u, v; \\ m_v & \text{if } i \neq j \text{ and } i, j \text{ have a common end } v; \\ 0 & \text{otherwise.} \end{cases}$$

Conversely, the Cartan matrix  $C^A$  of a Brauer tree algebra  $A$  and the data of extensions among simple  $A$ -modules determine the Brauer tree of  $A$  by the following:



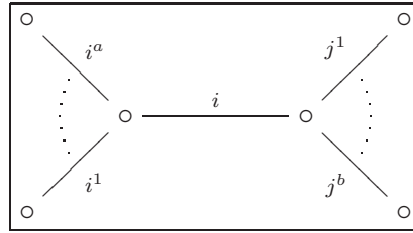
**Method 3.2.** Let  $A$  be a Brauer tree algebra. Assume that the Cartan matrix  $C^A$  of  $A$  and  $\dim_k \text{Ext}_A^1(S_i, S_j)$  for any  $i, j \in E$  are given. We explicitly determine the Brauer tree  $G$  of  $A$ .

(1) We give the cyclic ordering containing each edge. Fix any  $i \in E$ . We define a subset  $I$  of  $E$  by  $I = \{j \in E \mid C_{ij}^A \neq 0\}$ . Since  $G$  is a Brauer tree, one has a disjoint union  $I = \{i\} \cup I_0 \cup I_1$  satisfying  $C_{i_0 i_1}^A = 0$  for any  $i_0 \in I_0$  and  $i_1 \in I_1$ . Moreover, for any  $\ell \in \{0, 1\}$  and any  $j \in \{i\} \cup I_\ell$  there exists uniquely  $j' \in \{i\} \cup I_\ell$  such that  $\text{Ext}_A^1(S_j, S_{j'}) \neq 0$ . Thus we can take sequences

$$i = i^0, i^1, \dots, i^a, i^{a+1} = i \text{ in } \{i\} \cup I_0$$

$$i = j^0, j^1, \dots, j^b, j^{b+1} = i \text{ in } \{i\} \cup I_1$$

such that  $\text{Ext}_A^1(S_{i^x}, S_{i^{x+1}}) \neq 0$  for any  $0 \leq x \leq a$  and  $\text{Ext}_A^1(S_{j^y}, S_{j^{y+1}}) \neq 0$  for any  $0 \leq y \leq b$ . Hence we can explicitly determine the cyclic ordering containing  $i$ :



(2) We give the position of the exceptional vertex if it exists. Note that the exceptional vertex exists if and only if there is  $i \in E$  satisfying  $C_{ii}^A > 2$ . Put  $\mathcal{E} := \{i \in E \mid C_{ii}^A > 2\}$  and assume that  $\mathcal{E}$  is not an empty set. Since the Brauer tree  $G$  has only one exceptional vertex, we observe that all edges in  $\mathcal{E}$  have a common vertex  $v$  and any edge having the vertex  $v$  belongs to  $\mathcal{E}$ . Thus the vertex  $v$  is exceptional.  $\square$

We show the following easy observation.

**Proposition 3.3.** *Let  $A$  and  $B$  be derived equivalent symmetric  $k$ -algebras. If  $A$  is a Brauer tree algebra, then so is  $B$ .*

*Proof.* By Theorem 1.5,  $A$  is derived equivalent to a Brauer tree algebra  $C$  for a star with the exceptional vertex in the center if it exists. Since  $A$  and  $B$  are derived equivalent, it follows that  $B$  and  $C$  are also derived equivalent. This implies that  $B$  is stable equivalent to  $C$ . Note that  $C$  is a symmetric Nakayama algebra. Hence the assertion follows from [ARS, X, Theorem 3.14].  $\square$

We also need the following result.

**Lemma 3.4.** [Ok, Lemma 2.1] *Let  $E_0$  be a subset of  $E$  and put  $e := \sum_{i \in E_0} e_i$ . Let  $T := T(E_0)$  be the Okuyama-Rickard complex with respect to  $E_0$ . Assume that  $A$  is a symmetric algebra. Now the endomorphism algebra  $B = \text{End}_{\mathbf{K}^b(\text{proj-}A)}(T)$  of  $T$  is stable equivalent to  $A$  and we denote the stable equivalence by  $F : \underline{\text{mod-}}A \xrightarrow{\sim} \underline{\text{mod-}}B$ . Then the following hold:*

- (1) *If  $i \notin E_0$ , then  $F(\Omega(S_i))$  is a simple  $B$ -module;*
- (2) *If  $i \in E_0$ , then  $F(Y_i)$  is a simple  $B$ -module, where  $Y_i$  is maximal amongst submodules of  $P_i$  such that any  $S_j$  ( $j \in E_0$ ) is not a composition factor of  $Y_i/S_i$ .*

Now we are ready to prove Theorem 2.2.

**Proof of Theorem 2.2.** Let  $G$  be the Brauer tree of  $A$  and we use the notation of Definition 2.1. Since the Okuyama-Rickard complex  $T := T(E \setminus \{i\})$  is tilting by Proposition 1.2,  $B := \mu_i(A)$  is a Brauer tree algebra by Proposition 3.3. Our goal is to show that the Brauer tree of  $B$  coincides with  $\mu_i(G)$ . To do this, we only have to calculate the Cartan matrix of  $B$  and extensions among simple  $B$ -modules.

Recall that  $T$  is defined as the direct sum of the following complexes:

$$\begin{array}{ccc} & (0\text{th}) & (1\text{st}) \\ T_i : & P_{i_1} \oplus P_{j_1} & \longrightarrow P_i \\ T_\ell : & P_\ell & \longrightarrow 0 \quad (\ell \neq i) \end{array}$$

(If the edge  $i$  is external, then replace the above first complex with  $P_{i_1} \rightarrow P_i$  or  $P_{j_1} \rightarrow P_i$ .)

(1) Let  $C^A, C^B$  be Cartan matrices of  $A, B$ . We calculate  $C_{\ell m}^B$ . For each  $\ell \in E$ , we denote by  $P_\ell^B$  a projective indecomposable  $B$ -module corresponding to  $T_\ell$ .

- (i) We can easily check  $C_{\ell m}^B = C_{\ell m}^A$  for any  $\ell \neq i$  and  $m \neq i$ .
- (ii) We calculate  $C_{i\ell}^B$  for  $\ell \neq i$ . If  $\ell \neq i$ , then we have equalities

$$\begin{aligned} C_{i\ell}^B &= \dim_k \text{Hom}_B(P_i^B, P_\ell^B) \\ &= \dim_k \text{Hom}_{\mathbf{K}^b(\text{proj-}A)}(T_i, T_\ell) \\ &= \dim_k \text{Hom}_A(P_{i_1}, P_\ell) + \dim_k \text{Hom}_A(P_{j_1}, P_\ell) - \dim_k \text{Hom}_A(P_i, P_\ell) \\ &= C_{i_1\ell}^A + C_{j_1\ell}^A - C_{i\ell}^A. \end{aligned}$$

Therefore one sees the following:

$$C_{i\ell}^B = \begin{cases} 0 & (\ell \in \{i_2, \dots, i_a\} \text{ or } \{j_2, \dots, j_b\}); \\ C_{i_1\ell}^A \neq 0 & (\ell \in \{g_1, \dots, g_c\}); \\ C_{j_1\ell}^A \neq 0 & (\ell \in \{h_1, \dots, h_d\}); \\ m_{v(\ell)} & (\ell = i_1 \text{ or } j_1); \\ 0 & (\text{otherwise}) \end{cases}$$

where  $v(\ell)$  is the vertex of  $\ell$  that  $i$  does not have.

- (iii) We show  $C_{ii}^B = m_{v(i_1)} + m_{v(j_1)}$ . One sees equalities

$$\begin{aligned} C_{ii}^B &= \dim_k \text{Hom}_B(P_i^B, P_i^B) \\ &= \dim_k \text{Hom}_{\mathbf{K}^b(\text{proj-}A)}(T_i, T_i) \\ &= C_{ii}^A + C_{i_1i_1}^A + C_{j_1j_1}^A + 2C_{i_1j_1}^A - 2(C_{ii_1}^A + C_{ij_1}^A) \\ &= m_{v(i_1)} + m_{v(j_1)}. \end{aligned}$$

(2) For each  $\ell \in E$ , we put  $S_\ell^B = P_\ell^B / \text{rad} P_\ell^B$ . We calculate  $\text{Ext}_B^1(S_\ell^B, S_m^B)$ . We denote by  $F : \underline{\text{mod}}\text{-}A \rightarrow \underline{\text{mod}}\text{-}B$  the stable equivalence between  $A$  and  $B$  given by  $T$ . By Lemma

3.4, one sees that  $F$  sends

$$X_\ell := \begin{cases} \Omega(S_i) & (\ell = i) \\ Y_\ell & (\ell = i_1 \text{ or } j_1) \\ S_\ell & (\text{otherwise}) \end{cases}$$

to  $S_\ell^B$ , where  $Y_\ell$  is a unique submodule of  $P_\ell$  whose Loewy series is  $\begin{pmatrix} S_i \\ S_\ell \end{pmatrix}$ .

- (a) We can easily check  $\text{Ext}_B^1(S_\ell^B, S_m^B) \simeq \text{Ext}_A^1(S_\ell, S_m)$  for  $\ell, m \notin \{i, i_1, j_1\}$  or  $\ell = m = i$ .  
(b) We calculate  $\text{Ext}_B^1(S_\ell^B, S_m^B)$  for  $\ell \in \{i, i_1, j_1\}$  and  $m \notin \{i, i_1, j_1\}$ . One has isomorphisms

$$\begin{aligned} \text{Ext}_B^1(S_\ell^B, S_m^B) &\simeq \text{Ext}_A^1(X_\ell, S_m) \\ &\simeq \underline{\text{Hom}}_A(\Omega(X_\ell), S_m) \\ &\simeq \text{Hom}_A(\Omega(X_\ell), S_m) \\ &\simeq \begin{cases} \text{Hom}_A(\Omega^2(S_i), S_m) & (\ell = i) \\ \text{Hom}_A(\Omega(Y_\ell), S_m) & (\ell = i_1 \text{ or } j_1) \end{cases} \\ &\begin{cases} \neq 0 & (\ell = i \text{ and } m \in \{g_1, h_1\}) \\ \neq 0 & ((\ell, m) = (i_1, i_2) \text{ or } (j_1, j_2)) \\ = 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

Similarly, for  $\ell \notin \{i, i_1, j_1\}$  and  $m \in \{i, i_1, j_1\}$  we obtain isomorphisms

$$\begin{aligned} \text{Ext}_B^1(S_\ell^B, S_m^B) &\simeq \text{Hom}_A(S_\ell, \Omega^{-1}(X_m)) \\ &\simeq \begin{cases} \text{Hom}_A(S_\ell, S_i) & (m = i) \\ \text{Hom}_A(S_\ell, \Omega^{-1}(Y_m)) & (m = i_1 \text{ or } j_1) \end{cases} \\ &\begin{cases} \neq 0 & ((\ell, m) = (i_a, i_1), (g_c, i_1), (j_b, j_1) \text{ or } (h_d, j_1)) \\ = 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

- (c) We show  $\text{Ext}_B^1(S_\ell^B, S_i^B) \neq 0$  for  $\ell \in \{i_1, j_1\}$ . One sees isomorphisms

$$\begin{aligned} \text{Ext}_B^1(S_\ell^B, S_i^B) &\simeq \text{Ext}_A^1(Y_\ell, \Omega(S_i)) \\ &\simeq \underline{\text{Hom}}_A(Y_\ell, S_i) \\ &\simeq \text{Hom}_A(Y_\ell, S_i) \\ &\neq 0. \end{aligned}$$

Similarly, for  $\ell \in \{i_1, j_1\}$  we have an isomorphism

$$\begin{aligned} \text{Ext}_B^1(S_i^B, S_\ell^B) &\simeq \text{Hom}_A(S_i, \Omega^{-2}(Y_\ell)) \\ &\begin{cases} = 0 & \text{if } \ell \text{ is an internal edge} \\ \neq 0 & \text{otherwise} \end{cases} \end{aligned}$$

- (d) We calculate  $\text{Ext}_B^1(S_\ell^B, S_m^B)$  for  $\ell, m \in \{i_1, j_1\}$ . If  $\ell \neq m$ , it follows from (i) that  $C_{\ell m}^B = 0$ , which implies  $\text{Ext}_B^1(S_\ell^B, S_m^B) = 0$ . Let  $\ell = m$ . Since  $\text{Hom}_A(Y_\ell, Y_\ell) \simeq \text{Hom}_A(P_i, Y_\ell)$ , we have isomorphisms

$$\begin{aligned} \text{Ext}_B^1(S_\ell^B, S_\ell^B) &\simeq \text{Ext}_A^1(Y_\ell, Y_\ell) \\ &\simeq \text{Hom}_A(\Omega(Y_\ell), Y_\ell) \\ &\begin{cases} \neq 0 & \text{if } \text{Ext}_A^1(S_\ell, S_i) \neq 0 \text{ and } m_v > 1; \\ = 0 & \text{otherwise} \end{cases} \end{aligned}$$

where  $v$  is the common vertex of  $\ell$  and  $i$ .

Applying Method 3.2, we conclude that the Brauer tree of  $B$  is given by  $\mu_i(G)$ .  $\square$

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